# Technical Notes

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## An Approximate Analysis Technique for Design Calculations

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### 1 Introduction

In the design of complex structures, it is often necessary or desirable to employ approximations in the analysis to reduce computational cost and required computer storage. In automated or computer assisted design applications, it is often necessary to analyze a considerable number of designs, and it is the computational cost of these analyses that inhibits applications in many cases. Although no specific optimization problem is formulated in this Note, the method proposed is particularly applicable to such problems.

In this Note, a simple method is proposed with which one can obtain approximate results for analyses of modified designs based upon a limited number of exact analysis results. The idea of this method is based on the practically experienced fact that the number of design variables are usually far smaller than the degrees of freedom of the system, and the further observation that often large numbers of analysis degrees of freedom are dictated by the topology of the design rather than by the expected complexity of its behavior.

Some encouraging numerical examples, computed for space truss structures, are presented.

### 2 Approximate Method

In the static analysis of a structure using the displacement method, the system is expressed in the form

$$\underset{n\times n}{\mathbf{K}}\ \underset{n\times 1}{\mathbf{X}} = \underset{n\times 1}{\mathbf{P}} \tag{1}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{X}$ is the vector of displacement degrees of freedom, and $\mathbf{P}$ is the load vector. If the structure has $t$ design variables $(d_1,d_2, \ldots d_t)$, we can consider this set as a vector $\mathbf{D}$ in $t$-dimensional space. In this design space, consider the $r$ "basic" designs given by a set of the design vectors $\mathbf{D}_1,\mathbf{D}_2, \ldots \mathbf{D}_r$. Corresponding to these basic design vectors, the stiffness matrices $\mathbf{K}_1,\mathbf{K}_2, \ldots \mathbf{K}_r$ are obtained, and by solving the $r$ sets of simultaneous equations,

$$\mathbf{K}_i\mathbf{X}_i = \mathbf{P},\ i = 1,2, \ldots, r \tag{2}$$

basic displacement vectors $\mathbf{X}_1,\mathbf{X}_2, \ldots \mathbf{X}_r$ are computed.

**Table 1 Basic designs**

| Member | Design 1 | Design 2 | Design 3 | Design 4 | Design 5 |
|---|---|---|---|---|---|
| 1 ~ 18 | 1.0in.² | 0.2in.² | 1.0in.² | 1.0in.² | 1.0in.² |
| 19 ~ 42 | 1.0 | 1.0 | 0.3 | 1.0 | 1.0 |
| 43 ~ 72 | 2.0 | 2.0 | 2.0 | 0.4 | 2.0 |
| 73 ~ 124 | 0.5 | 0.5 | 0.5 | 0.5 | 0.1 |

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Consider now a new design vector $\mathbf{D}_N$ in the neighborhood of the basic design vectors. The stiffness matrix corresponding to $\mathbf{D}_N$ can be computed as $\mathbf{K}_N$, and the exact displacement vector due to the external load $\mathbf{P}$ would ordinarily be obtained by solving a set of simultaneous equations:

$$\mathbf{K}_N\mathbf{X}_N = \mathbf{P} \tag{3}$$

Here it is assumed that $\mathbf{X}_N$ can be approximated by the linear combination of basic displacement vectors $\mathbf{X}_1,\mathbf{X}_2, \ldots \mathbf{X}_r$.

$$\mathbf{X}_N \simeq \tilde{\mathbf{X}}_N = y_1\mathbf{X}_1 + y_2\mathbf{X}_2 + \ldots + y_r\mathbf{X}_r \tag{4}$$

If $\mathbf{X}_1,\mathbf{X}_2, \ldots \mathbf{X}_r$ are linearly independent and $r = n$, $\mathbf{X}_N$ can be the exact solution, but we are now considering a case for $r \ll n$. In matrix form, Eq. (4) will be expressed

$$\tilde{\mathbf{X}}_N = \mathbf{T}\mathbf{Y} \tag{5}$$

where

$$\underset{n\times r}{\mathbf{T}} = \left[\mathbf{X}_1,\mathbf{X}_2, \ldots \mathbf{X}_r\right]$$

$$\underset{r\times 1}{\mathbf{Y}} = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_r \end{Bmatrix}$$

The vector $\mathbf{Y}$ is then determined by solving a small system of Eqs. (6), which is obtained by substituting Eq. (5) into Eq. (3) assuming $\tilde{\mathbf{X}}_N \cong \mathbf{X}_N$ and premultiplying Eq. (3) by $\mathbf{T}^T$, i.e.,

$$\underset{r\times n}{\mathbf{T}^T}\ \underset{n\times n}{\mathbf{K}_N}\ \underset{n\times r}{\mathbf{T}}\ \underset{r\times 1}{\mathbf{Y}} = \underset{r\times n}{\mathbf{T}^T}\ \underset{n\times 1}{\mathbf{P}} \tag{6}$$

Introducing the notation

$$\mathbf{K}_R = \mathbf{T}^T\mathbf{K}_N\mathbf{T};\quad \mathbf{P}_R = \mathbf{T}^T\mathbf{P} \tag{7}$$

we have

$$\underset{r\times r}{\mathbf{K}_R}\ \underset{r\times 1}{\mathbf{Y}} = \underset{r\times 1}{\mathbf{P}_R} \tag{8}$$

In short, an approximate displacement vector $\tilde{\mathbf{X}}_N$ can be ob-

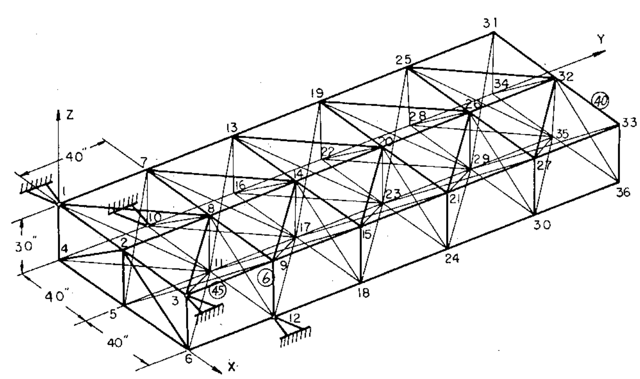


**Fig. 1 Example structure.**

**Table 2 Force and displacement results**

| Load condition | | Node & direction | Member | Design A Exact | Design A % Error | Design B Exact | Design B % Error | Design C Exact | Design C % Error | Design D Exact | Design D % Error |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Displacement | 21 x | ... | −0.00049 in. | 80.9 | 0.00026 in. | 304.9 | 0.00176 in. | 151.96 | 0.00203 in. | 112.4 |
| | | y | ... | 0.07379 | 0.095 | 0.04531 | −0.863 | 0.22048 | −0.443 | 0.29299 | −0.262 |
| | | z | ... | −0.70787 | −0.08 | −0.61341 | 0.101 | −2.06159 | 0.147 | −2.63290 | 0.114 |
| | | 33 x | ... | 0.02058 | 1.30 | 0.00699 | −5.354 | 0.03504 | −4.135 | 0.05232 | −2.720 |
| | | y | ... | 0.08946 | 0.11 | 0.05502 | −0.932 | 0.26764 | −0.459 | 0.35557 | −0.266 |
| | | z | ... | −1.49886 | $0.5 \times 10^{-4}$ | −1.26867 | −0.040 | −4.33951 | −0.026 | −5.56260 | −0.009 |
| | Force | ... | 6 | −8845.6 lb | 0.114 | −9109.7 lb | −1.892 | −8920.3 lb | −0.296 | −8892.9 lb | −0.065 |
| | | ... | 40 | 7719.8 | 1.283 | 7866.3 | −5.367 | 7884.7 | −4.128 | 7849.9 | −2.721 |
| | | ... | 45 | 34169.0 | 0.154 | 35259.4 | −0.881 | 34439.2 | −0.571 | 34257.5 | −0.351 |
| 2 | Displacement | 21 x | ... | −0.35566 in. | −0.069 | −0.30074 in. | 0.110 | −0.97534 in. | 0.115 | −1.25301 in. | 0.080 |
| | | y | ... | 0.01529 | 1.000 | 0.01164 | −5.069 | 0.04940 | −3.622 | 0.06349 | −2.204 |
| | | z | ... | 0.58594 | −0.044 | 0.51852 | 0.087 | 1.64329 | 0.098 | 2.09776 | 0.068 |
| | | 33 x | ... | −0.66149 | 0.005 | −0.56748 | −0.098 | −1.83123 | −0.095 | −2.34658 | −0.058 |
| | | y | ... | 0.00978 | 2.687 | 0.00864 | −10.988 | 0.03292 | −8.991 | 0.04127 | −5.647 |
| | | z | ... | 1.20350 | 0.001 | 1.06738 | −0.018 | 3.37584 | −0.015 | 4.30316 | −0.004 |
| | Force | ... | 6 | 5456.2 lb | −0.014 | 5156.8 lb | 1.706 | 5132.5 lb | −0.412 | 5197.6 lb | −0.289 |
| | | ... | 40 | −7058.6 | 0.988 | −6959.4 | −3.921 | −7118.5 | −2.986 | −7122.1 | −2.040 |
| | | ... | 45 | 11907.4 | 0.187 | 13834.2 | −1.503 | 12441.8 | −0.919 | 12163.4 | −0.560 |
| 3 | Displacement | 31 x | ... | 0.60856 in. | −0.030 | 0.52134 in. | −0.088 | 1.71186 in. | −0.042 | 2.19312 | −0.004 |
| | | y | ... | 0.07512 | 0.196 | 0.04815 | −1.768 | 0.22594 | −0.925 | 0.29843 | −0.512 |
| | | z | ... | 0.17278 | 0.351 | 0.17960 | −0.978 | 0.41627 | −1.169 | 0.51742 | −0.747 |
| | | 33 x | ... | 0.65456 | 0.011 | 0.53705 | −0.146 | 1.79012 | −0.114 | 2.30995 | −0.063 |
| | | y | ... | 0.01404 | −1.064 | 0.00681 | 10.086 | 0.04121 | 4.604 | 0.05636 | 2.486 |
| | | z | ... | −1.69222 | −0.006 | −1.45525 | −0.012 | −4.79081 | −0.004 | −6.13234 | 0.008 |
| | Force | ... | 6 | −8526.1 lb | −0.084 | −8342.6 lb | −1.174 | −8247.2 lb | −0.524 | −8298.9 lb | −0.171 |
| | | ... | 40 | 17014.0 | 0.565 | 17376.4 | −1.689 | 17405.0 | −1.663 | 17328.6 | −1.211 |
| | | ... | 45 | −1345.6 | 4.987 | −2648.0 | −18.315 | −1468.0 | −19.161 | −1295.7 | −12.403 |

tained by solving the smaller system in Eq. (8) instead of computing the exact solution $\mathbf{X}_N$ by solving the large system in Eq. (3).

This approach is equivalent to applying the Ritz-Galerkin principle in the subspace spanned by the set of vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_r$. The assumed modes $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_r$ are good basis vectors because they are the solutions of similar sets of equations.

## 3 Example

A three-dimensional space truss (Fig. 1) with 124 members and 96 degrees-of-freedom has 4 design variables; they are the cross sectional areas of members which are parallel to the $x$, $y$, and $z$ axes, respectively, and the cross-sectional area of the diagonal members.

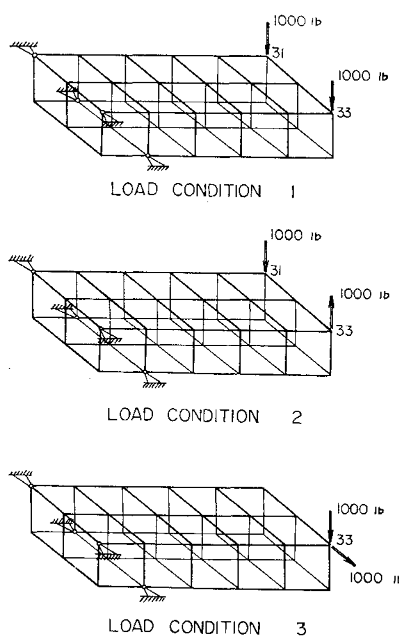


**Fig. 2 Load conditions.**

Five basic design points are taken as shown in Table 1. The load conditions considered are shown in Fig. 2.

Some force and displacement results for representative joints and members of the structure are given in Table 2. The exact answer is given along with the percentage error for the various loads and designs. This way of assessing the error suffers from the fact that the small components of displacement (e.g., design A, load condition 1, node 21, direction $x$) or small forces (e.g., design B, load condition 3, member 45) show large percentage error, even though the error is small absolutely. In spite of this, the results seem to be quite reasonable considering that this is a 5-degree-of-freedom approximation of a 96-degree-of-freedom system. The quantities of engineering significance are all accurate to within a few tenths of a percent.

Among the four new designs shown in Table 3, designs A and B are inside of the parallelopiped in the four-dimensional space having the 5 basic design points at 5 of itsc orners. design C is at the sixth corner of the parallelopiped, and design D is outside of it. Reducing the problem size is, of course, not without its costs. Some time must be spent in performing the products of Eq. (7). A computation time summary for this problem on the UNIVAC 1108 is given in Table 4.

## 4 Discussion

The approximate method discussed here will be particularly useful in automated optimization procedures because it can reduce the large computational overburden that conventional analysis methods entail. The choice of basic designs can, in this context, be more or less automatic. The final designs

**Table 3 New designs**

| Member | Design A | Design B | Design C | Design D |
|---|---|---|---|---|
| 1 ∼ 18 | 0.6 in.² | 1.5 in.² | 0.2 in.² | 0.15 in.² |
| 19 ∼ 42 | 0.5 | 1.5 | 0.3 | 0.2 |
| 43 ∼ 72 | 1.2 | 2.0 | 0.4 | 0.3 |
| 73 ∼ 124 | 0.3 | 0.3 | 0.1 | 0.08 |

**Table 4 Computation time summary**

| | |
|---|---|
| To solve Eq. (3) directly by Gauss elimination | 5.454 sec |
| To solve with approximate method | 0.653 sec |
| Details of approximate method | |
| to reduce stiffness matrix | 0.619 sec |
| to reduce load vector | 0.026 sec |
| to solve reduced equation | 0.004 sec |
| to assemble solution | 0.004 sec |

of various stages of the design sequence can be used as basic designs. There is no theoretical reason why there must be as many basic designs as there are design variables; although there is a certain intuitive appeal to having these numbers on the same order.

In many optimization applications, the most time-consuming part consists of a one-dimensional minimization, in which a number of analyses may be necessary along a "line" in the design hyperspace, and the useful basic design sets can obviously be restricted to points on this line.

It should be noted that the matrix products indicated in Eq. (7) can be performed without the complete assembly of the $\mathbf{K}_N$ matrix. This can be done in the case of finite element models with an element-by-element technique. In other cases the usually sparse matrices of structural analysis can be multiplied by matrix manipulation routines which take advantage of the zeros. None of these techniques were utilized in obtaining the results presented here.

# Calculation of Theoretical Equilibrium Nozzle Throat Conditions When Velocity of Sound is Discontinuous

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## 1. Introduction

It is generally a routine matter to determine theoretical throat conditions (such as temperature ratio and pressure ratio) in a De Laval nozzle assuming one-dimensional, isentropic, and equilibrium expansion.[1] However, the solution of throat conditions may be considerably more difficult when two condensed phases of the same species exist simultaneously in the vicinity of the throat. The difficulty arises due to a discontinuity in the velocity of sound which occurs when, during expansion, there is a transition from one condensed phase of a species to two condensed phases of that species. If this discontinuity exists in the vicinity of the throat, then usual procedures for determining throat conditions may either give inaccurate results or become impossible to use.

This Note first presents a brief summary of the equations and procedures which may be used to determine throat conditions for usual thermodynamic conditions. Then the special case which involves discontinuities is discussed and some new procedures are presented for obtaining throat conditions for this special case.

## 2. Usual Procedures for Obtaining Throat Conditions

For the case of a perfect gas with constant specific heat ratio $\gamma$, simple expressions exist for obtaining temperature ratio and pressure ratio at the throat.[2] For the case of a dissociating gas, where $\gamma$ is not constant, throat conditions require considerably more calculations. Equilibrium compositions and thermodynamic and flow properties such as temperature, velocity, and area per unit mass flow rate must be determined for several pressures in the vicinity of the throat.[1] Then, by interpolation or iteration, the pressure must be found for which area is minimum or, equivalently, for which the velocity of flow equals the velocity of sound (Mach number = 1).

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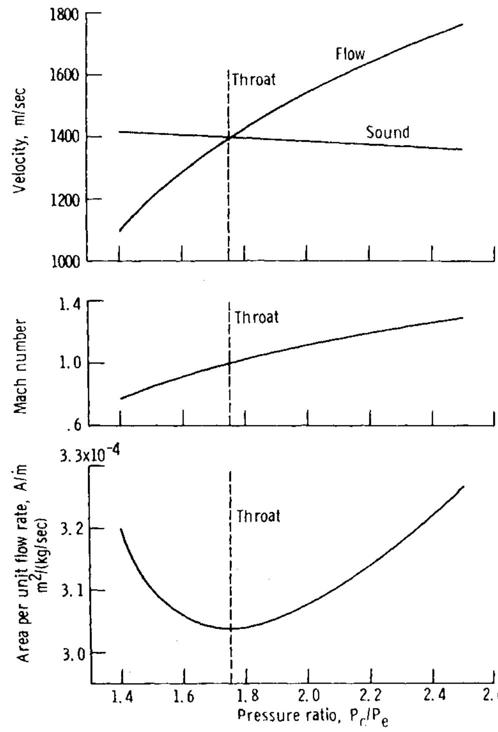


**Fig. 1 Some parameters as functions of pressure ratio in vicinity of nozzle throat for typical rocket propellant.**

The two methods by which throat conditions may be determined are illustrated in Fig. 1. This figure shows curves of the velocity of flow $u$, velocity of sound $a$, Mach number, and area/unit mass flow rate $A/\dot{m}$ that were obtained from theoretical equilibrium calculations of a typical rocket propellant. It is clear that throat occurs where $u = a$ (Mach number = 1) or where $A/\dot{m}$ is a minimum. This result depends on $A/\dot{m}$ and its derivative being continuous.

Equations for the determination of $u$, $a$, and $A/\dot{m}$ are given in Refs. 1 and 3. These equations, in somewhat different form, are

$$a_S = (\gamma_S nRT)^{1/2} \tag{1}$$

$$u = 2(h_c - h)^{1/2} \tag{2}$$

$$A/\dot{m} = 1/\rho u \tag{3}$$

where

$$\gamma_S \equiv (\partial \ln P/\partial \ln \rho)_S = -\gamma/(\partial \ln v/\partial \ln P)_T \tag{4}$$

$$\gamma \equiv c_P/c_V \tag{5}$$

and $n$ is the number of moles/unit mass (reciprocal of molecular weight), $R$ is a gas constant, $T$ is temperature, $h$ is enthalpy/unit mass, $\rho$ is density, $v$ is volume/unit mass, $c_P$ is the specific heat at constant pressure, and $c_V$ is the specific heat at constant volume. The subscript $c$ refers to combustion conditions and the subscript $S$ refers to constant entropy.

Equation (1) is obtained by combining the expression for the ideal equation of state

$$P = \rho nRT \tag{6}$$